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Simple spirals on double Warsaw circles

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Abstract

Let D be a symmetric double Warsaw circle in the plane E^2 and let r be the antipodal rotation of D. In [Fund. Math. 144 (1994) 1–9], Mańka proved there is a spiral P in E^2 limiting on D such that r extends to a fixed-point-free homeomorphism of the continuum $D \cup P$ into itself. It follows that $D \cup P$ lies in a uniquely arcwise connected continuum without the fixed-point property. Mańka's spiral P has an increasing amplitude condition that distinguishes it from simple spirals. Suppose S is a simple spiral in Euclidean 3-space E^3 limiting on D and a map f of $D \cup S$ into $D \cup S$ is an extension of r. We prove that f has a fixed point. This theorem remains true when r is replaced by any period 2 homeomorphism of D onto D. However, there exist a homeomorphism g of D onto itself and a simple spiral T in E^2 limiting on D such that g extends to a fixed-point-free homeomorphism h of $D \cup T$ into itself. We use the extension h to show a uniquely arcwise connected continuum defined by Holsztyński [Fund. Math. 64 (1969) 289–312] does not have the fixed-point property.

MSC: 54F15; 54H25

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1. Introduction

A space X has the *fixed-point property* if each map of X into X has a fixed point. A *continuum* is a nondegenerate compact connected metric space. A set is *uniquely arcwise connected* if it is arcwise connected and does not contain a simple closed curve.

In 1960, Young [11] described a uniquely arcwise connected continuum Y in E^3 that does not have the fixed-point property. Holsztyński [5] in 1969 formally defined a uniquely

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arcwise connected continuum that is very similar (maybe homeomorphic) to Y. Also in 1969, Bing [1] defined a uniquely arcwise connected continuum B with the fixed-point property and a disk C such that $B \cap C$ is an arc and $B \cup C$ admits a fixed-point-free map. In 1986, Mohler and Oversteegen [9] used Y to define two uniquely arcwise connected continua, the first admitting a fixed-point-free map that is monotone, and the second admitting one that is open.

Several fixed-point theorems have been established for uniquely arcwise connected continua. Mohler [8] in 1975 proved that every homeomorphism of a uniquely arcwise connected continuum into itself has a fixed point. This result is generalized to local homeomorphisms in [9]. In 1979, Hagopian [3] proved every uniquely arcwise connected plane continuum has the fixed-point property. Also see [2,4,6,10,12].

Let γ be the map of $(0,1]$ onto $[1,3]$ defined by $\gamma(x) = 2 + \sin\frac{\pi}{x}$ and let Γ denote the graph of γ in E^2, that is, $\Gamma = \{(x, \gamma(x)) \in E^2 \mid x \in (0,1]\}$. Define in E^2 points $a = (0, -3)$, $b = (2, 0)$, and $d = (1, 2)$.

Notation. For each two points p and q of E^2, we denote the straight line interval with endpoints p and q by $\overline{pq}$.

The set $A = \overline{ab} \cup \overline{bd} \cup \Gamma$ is a ray (a topological half-line) with endpoint a that limits on the interval $\overline{(0,1)(0,3)}$. Note that $\mathrm{Cl}\, A$ (the closure of A) is a topologist's sine curve.

We assume that E^2 is the xy-plane in E^3.

Let α be the map of E^3 onto E^3 defined by $\alpha(x, y, z) = (-x, -y, -z)$.

Let D be the continuum $\mathrm{Cl}(A \cup \alpha(A))$ in E^2. Every continuum homeomorphic to D is called a *double Warsaw circle.*

Let r be the restriction of α to D. Note that r is the antipodal rotation of D.

In [11,1,5], fixed-point-free maps are defined on uniquely arcwise connected continua by extending antipodal rotations of double Warsaw circles to limiting spirals. The spirals in [1,5] are called simple because their oscillating parts have constant amplitude. Whether or not the spiral in [11] is simple cannot be determined from its brief description. In 1994, Mańka [7] defined a class of more complicated planar spirals limiting on D and proved the closure of each of these spirals admits a fixed-point-free extension of r. Theorem 3.1 (below) shows no simple spiral limiting on D has this property. Hence the examples of [11, 1,5] should be revised. Replacing the spirals in these continua with Mańka's spirals is one way to eliminate the difficulty. However, Mańka's spirals are intricate. A more natural remedy is to extend a map of D other than r to a simple spiral. We define two maps of D with the required extension property in Section 4 (below).

2. Preliminaries

A *chain* in a space X is a finite collection $\mathcal{C} = \{L_1, L_2, \ldots, L_n\}$ of open sets in X such that $L_i \cap L_j \neq \emptyset$ if and only if $|i - j| \leqslant 1$. If $n > 2$ and L_1 also intersects L_n, the collection $\mathcal{C}$ is called a *circular chain* in X. Each L_i is called a *link* of $\mathcal{C}$. Two links L_i and L_j of a chain or circular chain are *consecutive links* if $L_i \cap L_j \neq \emptyset$. For convenience, we let $\bigcup \mathcal{C}$ denote $\bigcup\{L_i \mid 1 \leqslant i \leqslant n\}$.

An arc E is *folded* in a chain $\mathcal{C} = \{L_1, L_2, \ldots, L_n\}$ if $E \subset \bigcup \mathcal{C}$, the endpoints of E are in $L_n \setminus L_{n-1}$, and E intersects L_1. An arc E is *simply folded* in $\mathcal{C}$ if

(1) E is folded in $\mathcal{C}$,
(2) $E \cap L_1$ is connected, and
(3) $E \cap L_i$ $(1 < i \leqslant n)$ has exactly two components.

A *ray* is a set homeomorphic to $[0, 1)$.

An arc or ray F *runs straight* in a chain or circular chain $\mathcal{C}$ if $F \subset \bigcup \mathcal{C}$ and for each link L of $\mathcal{C}$, no component of $F \cap L$ has two endpoints in an element of $\mathcal{C}$.

Let

$$\mathcal{C} = \bigl\{L_1, \ldots, L_i, \ldots, L_j, \ldots, L_k, \alpha(L_1), \ldots, \alpha(L_i), \ldots, \alpha(L_j), \ldots, \alpha(L_k)\bigr\}$$

be a circular chain of open 3-cells with diameters less than $\frac{1}{4}$ in E^3 covering D such that $b \in L_1$, the closures of non-consecutive links are disjoint, L_i is the only link of $\mathcal{C}$ that contains $(0, 1)$, and L_j is the only link of $\mathcal{C}$ that contains $(0, 3)$. The circular chain $\mathcal{C}$ is *acceptable* if for each integer n such that $1 \leqslant n < i$ or $j < n \leqslant k$, the set $D \cap L_n$ is connected.

Let $\Phi = \{L_i, \ldots, L_j\}$ and $\Omega = \{\alpha(L_i), \ldots, \alpha(L_j)\}$. Note that $\overline{(0,1)(0,3)} \subset \bigcup \Phi$ and $\overline{(0,-1)(0,-3)} \subset \bigcup \Omega$.

Let S be a ray and let $\prec$ be the linear order of S induced by $<$ from $[0, 1)$. For each point x of S, let $S(x)$ denote the ray $\{y \in S \mid x \preceq y\}$.

A ray S is a *spiral* limiting on D if for each positive number ε there exist an acceptable circular chain $\mathcal{C}$ with mesh less than ε and point $x(\varepsilon)$ of $S \cap L_1$ such that

(2.1) $S(x(\varepsilon))$ runs straight in the circular chain

$$\begin{aligned}\mathcal{C}' = \Bigl\{&L_1, \ldots, L_{i-1}, \bigcup \Phi, L_{j+1}, \ldots, L_k,\\ &\alpha(L_1), \ldots, \alpha(L_{i-1}), \bigcup \Omega, \alpha(L_{j+1}), \ldots, \alpha(L_k)\Bigr\}.\end{aligned}$$

Note that $D \subset \mathrm{Cl}(S)$ if S is a spiral limiting on D.

Notation. Let Z be a set with uniquely arcwise connected arc-components. For each two points p and q in an arc-component of Z, we denote the arc in Z from p to q by $Z[p, q]$.

A spiral S limiting on D *runs counterclockwise* in $\mathcal{C}'$ if $S \subset \bigcup \mathcal{C}'$ and there exists an increasing sequence $\{y_n\}$ $(y_1 \prec y_2 \prec y_3 \prec \cdots)$ of points of S such that $S = \bigcup \{S[y_n, y_{n+1}] \mid n = 1, 2, \ldots\}$ and for each positive integer n, (1) $y_{2n-1} \in L_1$, (2) $y_{2n} \in L_k$, (3) $S[y_{4n-3}, y_{4n-2}] \subset \bigcup \{L_i \mid 1 \leqslant i \leqslant k\}$, and (4) $S[y_{4n-1}, y_{4n}] \subset \bigcup \{\alpha(L_i) \mid 1 \leqslant i \leqslant k\}$. We say that S *runs clockwise* in $\mathcal{C}'$ if S starts in L_1 and moves in a similar manner in the opposite direction in $\mathcal{C}'$. A spiral S limiting on D *eventually runs counterclockwise* (*clockwise*) in $\mathcal{C}'$ if there is a point x of S such that $S(x)$ runs counterclockwise (clockwise) in $\mathcal{C}'$.

A spiral S limiting on D is *simple* if for each positive number ε there exist an acceptable circular chain $\mathcal{C}$ with mesh less than ε and a point $x(\varepsilon)$ of $S \cap L_1$ satisfying (2.1) such that if H is a component of $S(x(\varepsilon)) \cap \bigcup(\Phi \setminus \{L_j\})$, $(S(x(\varepsilon)) \cap \bigcup(\Omega \setminus \{\alpha(L_j)\}))$, then either

(2.2) the arc $\mathrm{Cl}\,H$ has one endpoint in L_{i-1} $(\alpha(L_{i-1}))$ and the other in L_j $(\alpha(L_j))$ and the intersection of $\mathrm{Cl}\,H$ with each link of Φ (Ω) is connected, or

(2.3) both endpoints of $\mathrm{Cl}\,H$ are in L_j $(\alpha(L_j))$ and $\mathrm{Cl}\,H$ is simply folded in Φ (Ω).

3. A fixed-point theorem

Theorem 3.1. *Let S be a simple spiral in E^3 limiting on the double Warsaw circle D. Let f be a map of $D \cup S$ into $D \cup S$ that is an extension of the antipodal rotation r of D. Then f has a fixed point.*

Proof. Assume f moves each point of $D \cup S$.

Note that

(3.2) $f(S) \subset S$.

To see this assume the contrary. Then $f(S) \cap D \neq \emptyset$. By the continuity of f, one arc-component K of D contains $f(S)$. It also follows from the continuity of f that $f(D \cup U) \subset \mathrm{Cl}\,K$. However, $\mathrm{Cl}\,K$ is a proper subcontinuum of D and $f(D) = r(D) = D$. This contradiction establishes (3.2).

Let p be the first point of S with respect to $\prec$.

For each point z of S,

(3.3) $z \prec f(z)$.

For assume the contrary. Then by (3.2), $f(z) \prec z$. Since f is continuous and does not have a fixed point, it follows that $f(y) \prec y$ for each point y of $S[p,z]$. This contradicts the fact that p is the first point of S. Hence (3.3) is true.

Let

$$\mathcal{C} = \{L_1, \dots, L_i, \dots, L_j, \dots, L_k, \alpha(L_1), \dots, \alpha(L_i), \dots, \alpha(L_j), \dots, \alpha(L_k)\}$$

be an acceptable circular chain covering D as defined in Section 2 with $\varepsilon = \frac{1}{4}$ such that (2.1)–(2.3) are satisfied.

Note that S must eventually run counterclockwise or clockwise in $\mathcal{C}'$. We proceed with the assumption that S eventually runs counterclockwise in $\mathcal{C}'$. We omit the proof for the clockwise case, since it is similar to the one presented here for the counterclockwise case.

By (3.3), we can assume without loss of generality that S (itself) runs counterclockwise in $\mathcal{C}'$.

Let x_1 be a point of $S \cap L_1$ with the properties of $x(\varepsilon)$ in (2.1)–(2.3) such that for each point y of $S(x_1)$,

(3.4) if $y \in L \in \mathcal{C}$, then $f(y)$ belongs to $\alpha(L)$ or a link of $\mathcal{C}$ that intersects $\alpha(L)$.

Let z_1 be a point of $S(x_1) \cap \alpha(L_1)$ such that

(3.5) $S[x_1, z_1] \cap \alpha(L_2) = \emptyset$.

Let $\mathcal{A}_1$ denote the set of components of $S[x_1, z_1] \cap L_j$. By (2.1)–(2.3), $\mathcal{A}_1$ is finite.

Let $|\mathcal{A}_1|$ denote the cardinality of $\mathcal{A}_1$. There are $|\mathcal{A}_1|$ components of $S[x_1, z_1] \cap \bigcup(\Phi \setminus \{L_j\})$, one satisfying (2.2) and all others satisfying (2.3).

By (3.2)–(3.5), there exist points $x_2 \in \alpha(L_1) \cap S(z_1)$ and $z_2 \in L_1 \cap S(z_1)$ with $x_2 \prec z_2$ such that

(3.6) $f(S[x_1, z_1] \cap \bigcup \Phi) \subset S[x_2, z_2]$, and

(3.7) $S[x_2, z_2] \cap L_2 = \emptyset$.

Let $\mathcal{A}_2$ be the set of components of $S[x_2, z_2] \cap \alpha(L_j)$. Let $\mathcal{B}$ be the set of components of $S[x_2, z_2] \cap \alpha(L_{j-1} \cup L_j \cup L_{j+1})$.

By (2.1)–(2.3), each element of $\mathcal{B}$ contains exactly one element of $\mathcal{A}_2$.

Hence

(3.8) $|\mathcal{A}_2| = |\mathcal{B}|$.

By (3.4) and (3.6),

(3.9) f sends each element of $\mathcal{A}_1$ into an element of $\mathcal{B}$.

Furthermore,

(3.10) each element of $\mathcal{B}$ contains the image of an element of $\mathcal{A}_1$ under f.

To establish (3.10), let B be an element of $\mathcal{B}$. Let v be a point of $B \cap \alpha(L_j)$. Let s and t be points of $S[x_1, z_1]$ in L_3 and L_{k-2}, respectively. By (3.4), $f(s) \in \alpha(L_2 \cup L_3 \cup L_4)$ and $f(t) \in \alpha(L_{k-3} \cup L_{k-2} \cup L_{k-1})$. Since $S[x_1, z_1]$ runs straight in the chain

$$\Big\{L_1, \ldots, L_{i-1}, \bigcup \Phi, L_{j+1}, \ldots, L_k, \alpha(L_1)\Big\},$$

it follows from (3.4) and (3.6) that $f(S[s, t]) \subset S[x_2, z_2]$. Since $S[x_2, z_2]$ runs straight in

$$\Big\{\alpha(L_1), \ldots, \alpha(L_{i-1}), \bigcup \Omega, \alpha(L_{j+1}), \ldots, \alpha(L_k), L_1\Big\},$$

the sets $S[x_2, z_2] \setminus f(S[s, t])$ and $\alpha(L_{j-1} \cup L_j \cup L_{j+1})$ are disjoint. Since $f(S[s, t])$ is connected, there exists a point u of $S[s, t]$ such that $f(u) = v$. By (3.4), u belongs to $L_{j-1} \cup L_j \cup L_{j+1}$. Let M be the u-component of $(L_{j-1} \cup L_j \cup L_{j+1}) \cap S[s, t]$. By (2.1)–(2.3), $M \cap L_j$ is an element of $\mathcal{A}_1$. By (3.4), $f(M \cap L_j)$ is contained in an element of $\mathcal{B}$. By (2.1)–(2.3) and (3.4), $f(M)$ intersects only one element of $\mathcal{B}$. Since $f(u) \in B$, it follows that $f(M \cap L_j) \subset B$. Hence (3.10) is true.

By (3.9) and (3.10),

(3.11) $|\mathcal{B}| \leqslant |\mathcal{A}_1|$.

Hence, by (3.8) and (3.11),

(3.12) $|\mathcal{A}_2| \leqslant |\mathcal{A}_1|$.

Note that (3.12) implies $S[x_2, z_2]$ does not have more oscillations than $S[x_1, z_1]$.

Continuing this process, for each integer $n > 1$ we define points x_n and z_n of S with $x_n \prec z_n$ satisfying the following conditions.

If n is odd, then

(3.13) $x_n \in S(z_{n-1}) \cap L_1$ and $z_n \in S(z_{n-1}) \cap \alpha(L_1)$,

(3.14) $S[x_n, z_n] \subset \bigcup\{L_1, \ldots, L_k, \alpha(L_1)\}$,

(3.15) $\mathcal{A}_n$ denotes the set of components of $S[x_n, z_n] \cap L_j$, and

(3.16) $|\mathcal{A}_n| \leqslant |\mathcal{A}_{n-1}|$.

If n is even, then

(3.17) $x_n \in S(z_{n-1}) \cap \alpha(L_1)$ and $z_n \in S(z_{n-1}) \cap L_1$,

(3.18) $S[x_n, z_n] \subset \bigcup\{\alpha(L_1), \ldots, \alpha(L_k), L_1\}$,

(3.19) $\mathcal{A}_n$ denotes the set of components of $S[x_n, z_n] \cap \alpha(L_j)$, and

(3.20) $|\mathcal{A}_n| \leqslant |\mathcal{A}_{n-1}|$.

It follows from (3.16) and (3.20) that for every integer $n > 1$,

(3.21) $|\mathcal{A}_n| \leqslant |\mathcal{A}_1|$.

Let

$$\begin{aligned}\mathcal{E} = \big\{&V_1, \ldots, V_{i'}, \ldots, V_{j'}, \ldots, V_{k'},\\ &\alpha(V_1), \ldots, \alpha(V_{i'}), \ldots, \alpha(V_{j'}), \ldots, \alpha(V_{k'})\big\}\end{aligned}$$

be an acceptable circular chain covering D satisfying (2.1)–(2.3) that refines $\mathcal{C}$ with the property that $\{V_1, \ldots, V_{i'}\}$ has a subcollection $\mathcal{W}$ of cardinality $|\mathcal{A}_1| + 1$ such that $\bigcup \mathcal{W} \subset L_j$ and

(3.22) no component of $D \cap L_j$ intersects two elements of $\mathcal{W}$.

Here $b \in V_1$ and $V_{i'}$ is the only link of $\mathcal{E}$ that contains the point $(0, 1)$.

Let y be a point of $S(x_1) \cap V_1$ such that $S(y)$ runs straight in $\mathcal{E}'$ (the circular chain defined from $\mathcal{E}$ as $\mathcal{C}'$ was defined from $\mathcal{C}$). Let n be an odd integer such that $x_n \in S(y)$. The arc $S[x_n, z_n]$ intersects each element of $\mathcal{W}$. Each two elements of $\mathcal{W}$ can be joined by an arc in $D \cap \cup\Phi$. Since $\{V_1, \ldots, V_{k'}\}$ is a chain covering $(S[x_n, z_n] \cup D) \cap \bigcup \Phi$, it follows from (3.22) that no component of $S[x_n, z_n] \cap L_j$ intersects two elements of $\mathcal{W}$. Hence $|\mathcal{A}_n| > |\mathcal{A}_1|$. This contradiction of (3.21) completes the proof of Theorem 3.1. □

4. Fixed-point-free extensions

Let S be a spiral limiting on the double Warsaw circle D. Suppose ρ is a map of D onto D that extends to a fixed-point-free map f of $D \cup S$ into $D \cup S$. Then $D \cup S$ lies in a uniquley arcwise connected continuum X that admits a fixed-point-free extension of f [7, Theorem 4.1]. Hence X does not have the fixed-point property. If S is simple, it follows from the proof of Theorem 3.1 that ρ is not a period 2 homeomorphism.

A map σ of D onto D is *symmetric* if $\sigma(x, y) = (u, v)$ implies $\sigma(-x, -y) = (-u, -v)$ for each point (x, y) of D.

Example 4.1. There exists a symmetric homeomorphism g of D onto D and a simple spiral T in E^2 limiting on D such that g extends to a fixed-point-free homeomorphism h of $D \cup T$ into $D \cup T$.

To define g, let c denote $(\frac{3}{2}, 1)$ (the midpoint of $\overline{bd}$). For each positive integer n, let $\mu_n = \frac{2}{4n+1}$ and let p_n be the point of Γ with coordinates $(\mu_n, \gamma(\mu_n))$. Note that $\gamma(\mu_n) = 3$ for each n.

Let g be a symmetric homeomorphism of D onto D such that

$$g(p) = r(p) \text{ for every point } p \text{ of } D \text{ in the second quadrant of } E^2,$$
$$g(\overline{bc}) = r(\overline{bd}),$$
$$g(\overline{cd}) = r(\Gamma[d, p_1]),$$
$$g(\Gamma[d, p_1]) = r(\Gamma[p_1, p_2]), \quad \text{and}$$
$$g(\Gamma[p_n, p_{n+1}]) = r(\Gamma[p_{n+1}, p_{n+2}]) \quad \text{for each positive integer } n.$$

Define g so that each arc in $\overline{bd} \cup \Gamma$ that contains b is stretched by g onto a longer arc in $r(\overline{bd} \cup \Gamma)$ that has $r(b)$ as an endpoint. Since g is a symmetric homeomorphism, each arc in $r(\overline{bd} \cup \Gamma)$ containing $r(b)$ must be stretched in the same manner by g onto a longer arc in $\overline{bd} \cup \Gamma$ that has b as an endpoint.

We need the following notation to define the simple spiral T.

Notation. Given a set Z in E^2 and a real number ε, we let $Z \uparrow \varepsilon$ denote

$$\{(x, y) \in E^2 \mid (x, y - \varepsilon) \in Z\}.$$

For each positive integer n, let

$$A_n = \overline{(2 + n^{-1}, 0)(1, 2 + n^{-1})},$$
$$B_n = D[d, p_{2n-1}] \uparrow n^{-1},$$
$$C_n = \{(x, y) \in E^2 \mid 0 \leqslant x \leqslant \mu_{2n-1} \text{ and } y = 3 + n^{-1}\},$$
$$D_n = \overline{(0, 3 + n^{-1})(-2 - n^{-1}, 0)},$$
$$E_n = \alpha(D[d, p_{2n}]) \uparrow -n^{-1},$$
$$F_n = \{(x, y) \in E^2 \mid -\mu_{2n} \leqslant x \leqslant 0 \text{ and } y = -3 - n^{-1}\}, \quad \text{and}$$
$$G_n = \overline{(0, -3 - n^{-1})(2 + (n + 1)^{-1}, 0)}.$$

For each n, let

$$\Gamma_n = A_n \cup B_n \cup C_n \cup D_n \cup \alpha(A_n) \cup E_n \cup F_n \cup G_n.$$

Let $T = \bigcup_{n=1}^{\infty} \Gamma_n$. Note that T is a simple spiral limiting on D running counterclockwise in E^2 and one oscillating cycle is added each time T passes through either the first or the third quadrant.

There exists a natural fixed-point-free homeomorphism h of $D \cup T$ into $D \cup T$ that is an extension of g stretching each arc $A_n \cup B_n \cup C_n$ onto $\alpha(A_n) \cup E_n \cup F_n$ and each arc $\alpha(A_n) \cup E_n \cup F_n$ onto $A_{n+1} \cup B_{n+1} \cup C_{n+1}$.

By [7, Theorem 4.1], h extends to a fixed-point-free map of a uniquely arcwise connected continuum into itself.

The uniquely arcwise connected continuum H defined by Holsztyński [5, Eg. 7.2] has a subcontinuum K homeomorphic to $D \cup T$. Let ϕ be a homeomorphism that sends $D \cup T$ onto K. By Theorem 3.1, the antipodal rotation $\phi r \phi^{-1}$ of $\phi(D)$ cannot be extended to a fixed-point-free map of K into K. However, the symmetric homeomorphism $\phi g \phi^{-1}$ of $\phi(D)$ extends to the homeomorphism $\phi h \phi^{-1}$ of K into K and $\phi h \phi^{-1}$ extends to a fixed-point-free map of H onto H. Hence H does not have the fixed-point property.

Example 4.2. The symmetric homeomorphism g in Example 4.1 can be replaced by a symmetric map ζ of D onto D that agrees with r in the first quadrant and folds the arc $\overline{(0,3)(-2,0)}$ onto $\overline{(0,-1)(0,-3)} \cup \overline{(0,-3)(2,0)}$.

More precisely, ζ sends $\overline{(0,3)(-\frac{1}{2},\frac{9}{4})}$, $\overline{(-\frac{1}{2},\frac{9}{4})(-1,\frac{3}{2})}$, and $\overline{(-1,\frac{3}{2})(-2,0)}$ homeomorphically onto $\overline{(0,-3)(0,-1)}$, $\overline{(0,-1)(0,-3)}$, and $\overline{(0,-3),(2,0)}$, respectively.

Like g in Example 4.1, ζ extends to a fixed-point-free map ψ of $D \cup T$ into $D \cup T$. To define ψ, let q_n be the lowest point of $\Gamma[p_n, p_{n+1}]$ for each positive integer n.

The map ψ sends each arc $A_n \cup B_n \cup C_n \cup D_n$ onto $\alpha(A_n) \cup E_n \cup F_n \cup G_n$ and satisfies the following conditions.

For each positive integer n,

(1) ψ agrees with α on $A_n \cup B_n$,
(2) $\psi(C_n) = \overline{(-\mu_{2n-1}, -3-n^{-1})}$, and
(3) ψ sends $\overline{(0, 3+n^{-1})(-\frac{1}{2}, \frac{9}{4}+n^{-1})}$, $\overline{(-\frac{1}{2}, \frac{9}{4}+n^{-1})(-1, \frac{3}{2}+n^{-1})}$, and $\overline{(-1, \frac{3}{2}+n^{-1})(-2-n^{-1}, 0)}$ homeomorphically onto $\alpha(D[p_{2n-1}, q_{2n-1}] \uparrow n^{-1})$, $\alpha(D[q_{2n-1}, p_{2n}] \uparrow n^{-1})$, and $F_n \cup G_n$, respectively.

Similarly, ψ maps each arc $\alpha(A_n) \cup E_n \cup F_n \cup G_n$ onto $A_{n+1} \cup B_{n+1} \cup C_{n+1} \cup D_{n+1}$. Observe the ψ may be constructed so that it is continuous.

As in Example 4.1, the map $\phi \psi \phi^{-1}$ of K into K extends to a fixed-point-free map of H into H.

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